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Ideals of the Enveloping Algebra $U(sl_2)$

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In this paper we study the two-sided ideals of the enveloping algebra $U = U(sl_2(K))$ over an arbitrary field K of characteristic zero. Starting with two basic ideas, that an irreducible Lie module is generated by its highest weight vector and that the Lie module structure of U comes from its ring multiplication, we have found a "good" subset of U consisting of highest weight vectors for irreducible U-submodules of U so that each two-sided ideal of U is uniquely generated by at most two elements of that set. Actually, each ideal is generated as a two-sided ideal by just one element. By uniqueness, all the information about the ideal is encoded in the formula for its generator(s). For example, we can list and classify all the prime ideals by height, determine the intersection of an ideal with the center, find the radical ideals and the radical of an ideal, and determine when two ideals are included one in another. An interesting property is that each ideal of U can be uniquely written as a product of primes. We also obtain the "least common multiple" and the "greatest common divisor" formulas for the prime ideal factorizations of the intersection and the sum of two ideals. This paper contains many other results of this nature. © 1998 Academic Press

INTRODUCTION

This section contains a brief review of the general theory of Lie algebras and enveloping algebras that can be found in books like Dixmier [9], Humphreys [10], or Jacobson [11].

Let K be an arbitrary field. A *Lie algebra* over K is a vector space L over K together with a bracket operation $[\ ,\ ]$: $L \times L \to L$ which is bilinear in each argument and satisfies:

(i) $[a, a] = 0$ for all $a \in L$,

(ii) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all $a, b, c \in L$.

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The last relation is called the *Jacobi identity* for the Lie algebra L. Note that if $\operatorname{char}(K) \neq 2$, then (i) and (ii) are equivalent to

(i′) $[a,b] = -[b,a]$ for all $a, b \in L$,

(ii′) $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all $a, b, c \in L$.

Also, to check that a vector space with a bilinear bracket operation is a Lie algebra, it is enough to verify the axioms (i′) and (ii) on basis elements.

EXAMPLE. There are two main examples we use in this paper:

(1) Every associative algebra R is a Lie algebra via $[a,b] = ab - ba$ for all $a, b \in R$.

(2) $L = sl_2(K)$ is the three-dimensional Lie algebra with basis $\{x, y, z\}$ such that the bracket operation on basis elements is given by $[x,y] = z$, $[z,x] = 2x$ and $[z,y] = -2y$.

It is well known that any Lie algebra L can be embedded as a Lie algebra in a larger Lie algebra L' which is an associative algebra and whose Lie bracket operation is given as in the first of the previous examples.

EXAMPLE. It is not hard to see that the linear map $\phi\colon L = sl_2(K) \to M_2(K) = L'$ defined on basis elements by

$$\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \phi(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \phi(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

preserves the Lie bracket operation and gives an embedding of L in L' as a Lie algebra.

Also, for each Lie algebra L, there exists a maximal associative algebra $U(L)$ containing L as a Lie subalgebra and generated by L. The term "maximal" means that any other associative algebra containing L as a Lie subalgebra is uniquely a homomorphic image of $U(L)$. A way to construct the enveloping algebra $U = U(L)$ is given by the PBW theorem. If $\{x_1, x_2, x_3, \cdots\}$ is an ordered basis for L over K, then a basis for U over K is the set

$$\left\{ x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_s}^{a_s} \,\middle|\, s \geq 0, i_1 < i_2 < \cdots < i_s \right\}$$

of ordered monomials in the elements $x_1, x_2, x_3, \cdots$. For the case $L = sl_2(K)$, a basis for U over K is the set $\{x^a y^b z^c \mid a, b, c \geq 0\}$. The multiplication of two monomials of U is given by concatenation and can be reduced to an expression in terms of the basis elements, by means of the commuting relations:

$$xy - yx = [x, y] = z, \qquad zx - xz = [z, x] = 2x,$$
$$zy - yz = [z, y] = -2y.$$

Using the distributive law, we know how to multiply arbitrary elements of U.

EXAMPLE. In $U(sl_2(K))$, we have

$$\begin{aligned}(3xy - z)(y - x) &= 3xy^2 - 3xyx - zy + zx \\ &= 3xy^2 - 3x(xy - z) - (yz - 2y) + (xz + 2x) \\ &= 3xy^2 - 3x^2y + 4xz - yz + 2y + 2x.\end{aligned}$$

A K-vector space M is called a *(Lie) L-module* if there exists a bilinear map

$$L \times M \to M, \qquad (l, m) \to l \cdot m$$

so that

$$[a, b] \cdot m = a \cdot (b \cdot m) - b \cdot (a \cdot m)$$

for all $a, b \in L$ and $m \in M$. By (ii′), it is clear that L is itself an L-module via $l \cdot m = [l, m]$. It is not hard to see that any L-module M becomes a $U = U(L)$-module by putting

$$(x_{i_1} x_{i_2} \cdots x_{i_s}) \cdot m = x_{i_1} \cdot \Big(x_{i_2} \cdot \cdots \cdot \big(x_{i_{s-1}} \cdot (x_{i_s} \cdot m)\big) \cdots \Big)$$

and in this way M is both a Lie U-module and an associative U-module. Since L is embedded in $U(L)$ as a Lie algebra, by restriction, any Lie $U(L)$-module is also an L-module.

A K-subspace of an L-module M is called a *submodule* if it is closed under the action of L on M. The L-module M is called *irreducible* if it does not have any proper L-submodules and *completely reducible* if it is a direct sum of irreducible L-modules. As for the associative algebras, if L is a Lie subalgebra of L', then L' becomes an L-module. In particular, $U(L)$ is an L-module, hence a $U(L)$-module. Note that the $U(L)$-module structure on $U(L)$ coming from L is called the *adjoint representation* and is different from the right or left-module structure of the ring $U(L)$.

EXAMPLE. In $U = U(sl_2)$ we have

$$\begin{aligned}x \cdot (3y^2 - z) &= 3x \cdot y^2 - x \cdot z = 3(xy^2 - y^2x) - (xz - zx) \\ &= 3xy^2 - 3y^2x - (-2x) = 3xy^2 - 3y(xy - z) + 2x \\ &= 3xy^2 - 3(xy - z)y + 3yz + 2x \\ &= 3xy^2 - 3xy^2 + 3(yz - 2y) + 3yz + 2x \\ &= 6yz - 6y + 2x\end{aligned}$$

while

$$x(3y^2 - z) = 3xy^2 - 3xz.$$

In order to avoid any confusion, throughout this paper we write

$$\delta_x(f) = x \cdot f, \qquad \delta_y(f) = y \cdot f, \qquad \delta_z(f) = z \cdot f$$

for all $f \in U = U(sl_2)$. Also,

$$\delta_y^2 \delta_x \delta_z^3(f) = (y^2 x z^3) \cdot f = y \cdot y \cdot x \cdot z \cdot z \cdot z \cdot f.$$

The letter δ stands for "derivation," since L acts on $U = U(L)$ as derivations. Indeed, the product rule holds,

$$\delta_x(fg) = \delta_x(f)g + f\delta_x(g)$$

for all f, g and similarly for δ_y and δ_z.

From now on, unless otherwise stated, K is an arbitrary field of characteristic zero, $L = sl_2(K)$, and $U = U(L)$. It is well known that for each $m \geq 0$, there exist an $m + 1$-dimensional U-module $V(m)$. As a vector space, $V(m)$ has the basis $\{v_0, v_1, v_2, \dots, v_m\}$ and

$$\begin{cases} \delta_x(v_i) = (m - i + 1)v_{i-1} & \text{for } 1 \leq i \leq m, \qquad \delta_x(v_0) = 0 \\ \delta_y(v_i) = (i + 1)v_{i+1} & \text{for } 0 \leq i \leq m - 1, \qquad \delta_y(v_m) = 0 \\ \delta_z(v_i) = (m - 2i)v_i & \text{for } 0 \leq i \leq m. \end{cases} \tag{1}$$

Note that $V(m)$ splits up as a direct sum of $m + 1$ one-dimensional eigenspaces corresponding to the $m + 1$ distinct eigenvalues for z. The spaces $V_i = Kv_i$ are called *weight subspaces* of $V(m)$. The nonzero elements of V_i are called *weight vectors* of $V(m)$ corresponding to the weight (eigenvalue) $m - 2i$. Also, v_0 is called *the highest weight vector* of $V(m)$.

An important result about the enveloping algebra $U = U(sl_2)$ is that its center is a polynomial ring in the indeterminate $h = z^2 + 4xy - 2z$, hence it is a principal ideal domain. See [13, Section 6.5]. In this paper we will freely use the fact that in a PID, the ideal generated by two elements is the ideal generated by their greatest common divisor.

We close this introductory chapter with two important notations. If S is a subset of U, the U-module generated by S under the adjoint action will be denoted by $[S]$ and the two-sided ideal of U generated by S will be written (S). For $f \in U$, the symbols $[f]$ and (f) represent respectively the U-module and the two-sided ideal generated by the element f.

Section 1 starts with the module structure for $U = U(sl_2)$. In Theorem 1.4 we prove that every two-sided ideal of U is generated as ideal by finitely many of its highest weight elements. We shall see (Theorem 1.6) that we may assume these elements to have different weights and, by adding them up, we conclude in Corollary 1.7 that each ideal of U is generated as a two-sided ideal by just one element.

Theorem 2.2 is the key result in Section 2. It enables us in Theorem 2.3 to determine the module structure for ideals of the form (x^n), $n \geq 1$. In particular we can express their intersection with the center $Z(U)$ and obtain two properness conditions.

In the next section we reduce the number of generators in Theorem 1.6 to at most two (Theorem 3.2) and impose more conditions on them so that the expression is unique (Theorem 3.7). This is the main result of the paper and can be stated as follows:

THEOREM. *Let I be a proper ideal of U. Then there exist a unique integer $n \geq 0$ and monic polynomials P, Q with $Q(h) | f_{n0}(h)$ and $h - n^2 + 1 | Q(h)$, whenever $n > 0$, so that $I = (x^n P(h), P(h)Q(h))$.*

There are two equivalent statements for this result. One of them, Theorem 3.7″, reminds us that we can always add the two generators to get just one generator for each ideal of U. With the module structure for (x^n) given in Section 2, in Theorem 3.3 we are able to find the module structure of an arbitrary ideal $I = (x^n P(h), P(h)Q(h))$. Corollary 3.5 says that $I \cap Z(U) = P(h)Q(h)K[h]$. In particular, I is proper iff $PQ \neq 1$ and $I \cap Z(U) \neq 0$ for all $0 \neq I \triangleleft U$. The latter is true for any semisimple Lie algebra L (see Dixmier [9, Proposition 4.2.2]). The last part of the section contains some applications of the module structure given in Theorem 3.3. We determine the (unique) generators for the sum and intersection of two ideals and give the condition on generators equivalent to saying that two ideals are contained one in another. At the end we give a different proof to a property seen first by Bavula in [2, 3.2 and 4.6], that the lattice of two-sided ideals of U is distributive.

In Section 4 we determine the list of prime ideals of U and classify them by height. Basically we determine the expressions for P and Q equivalent to I being prime. For $K = C$, this list of prime ideals was first announced by Nouazé and Gabriel in [13] (see also Dixmier [8]).

More applications can be found in the last section. We prove that the product of finitely many maximal ideals of finite codimension equals their intersection, that each ideal can be uniquely written as a product of primes, and that a product of two ideals is commutative. The last two properties can be found with different proofs in Bavula [2]. Next we determine the prime ideal factorizations for the radical ideals, the radical of an ideal, and the sum and intersection of two ideals. At the end we prove that the intersection of an infinite set of distinct ideals of U is trivial.

This paper is a part of the author's Ph.D. thesis at the University of Wisconsin-Madison. The thesis ([3]) also includes analogous results for other Lie algebras, with particular interest in the enveloping algebra $U(sl_3)$ and tensor powers of $U(sl_2)$. See also [4] and [5].

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1. THE EXISTENCE

Let $U = U(L)$ be the enveloping algebra of the Lie algebra $L = sl_2(K)$ over an arbitrary field K of characteristic zero. Recall that $\{x, y, z\}$ is the standard basis for L with $[x, y] = z$, $[z, x] = 2x$, and $[z, y] = -2y$. Then by the PBW theorem a K-basis for U is the set

$$\{x^a y^b z^c \mid a, b, c \geq 0\}$$

of ordered monomials in x, y, z. We have seen in the Introduction that U is an L-module, hence a U-module. In this section we shall see that $U \cdot x^n = [x^n]$ is an irreducible U-submodule of U of dimension $2n + 1$ and if $h = z^2 + 4xy - 2z$ is the generator of the center of U, then

$$U = \oplus \sum_{i,j=0}^{\infty} [h^i x^j]$$

is an expression of U as a direct sum of irreducible U-modules. Therefore U is a completely reducible U-module. Now as the two-sided ideals of U are U-modules under the adjoint action, we obtain in particular that all the ideals of U are completely reducible U-modules and therefore they are direct sums of certain irreducible submodules of U. On the other hand, each irreducible U-module is generated by just one element. In particular it is generated by its highest weight vector. As a consequence from this, if we start with a two-sided ideal I of U, then I is generated as a U-module by the highest weight vectors of the irreducible summands of I coming from complete reducibility. Moreover, as the adjoint action of U on U comes from the ring multiplication in U, it follows that I is generated as a two-sided ideal by the highest weight vectors of the irreducible U-submodules of I. Furthermore, U being a Noetherian ring, the ideal I is finitely generated as a (one-sided, hence) two-sided ideal. It follows that I is generated as a two-sided ideal by finitely many highest weight elements. With this idea in mind we can start this first section.

Define U_n to be the set of elements of U of total degree $\leq n$. Then U_n is a finite dimensional vector space and it is well known that U_n is a completely reducible U-submodule of U. The following lemma provides an example of an irreducible U-submodule of U.

LEMMA 1.1. $[x^n] \cong V(2n)$

Proof. Recall that any finite dimensional irreducible U-module is isomorphic to $V(m)$ for some m. As the weight vectors inside U_n are linear combinations of monomials of U_n of the same weight, it is necessary to look at a generic monomial:

$$x^{\alpha} y^{\beta} z^{\gamma}, \qquad \alpha + \beta + \gamma \leq n.$$

The weight of such a monomial is determined by the eigenvalue of the action of z. Recall that

$$\delta_z(x) = 2x, \qquad \delta_z(y) = -2y, \qquad \delta_z(z) = 0$$

and therefore the weights for x, y, and z are 2, -2, and 0, respectively. In general, for a large product we use the fact that z acts as derivation and have

$$\begin{aligned} \delta_z\left(x^{\alpha} y^{\beta} z^{\gamma}\right) &= \delta_z(x) x^{\alpha-1} y^{\beta} z^{\gamma} + \cdots + x^{\alpha-1} \delta_z(x) y^{\beta} z^{\gamma} + \cdots \\ &\quad + x^{\alpha} y^{\beta} z^{\gamma-1} \delta_z(z) \\ &= (2\alpha - 2\beta) x^{\alpha} y^{\beta} z^{\gamma} \end{aligned}$$

and hence the weight is $2\alpha - 2\beta$. Deduce from here that all the weights in U_n are even, the maximal weight is $2n$, and it is attained exactly for one vector, x^n, that is, when $\alpha = n$, $\beta = 0$, and $\gamma = 0$. Also, the minimal weight is $-2n$ and it is attained for the monomial y^n, that is, for $\alpha = 0$, $\beta = n$, and $\gamma = 0$. Now as the highest weight subspace of U_n has dimension one and the highest weight is $2n$, U_n must have a unique irreducible U-submodule of dimension $2n + 1$, all other irreducible submodules having smaller dimension. Observe that the $2n + 1$-dimensional irreducible submodule is generated either by its highest weight vector x^n or by its lowest weight vector y^n and we have $[x^n] = [y^n]$. A basis for this module is the set

$$\left\{x^n, \delta_y(x^n), \delta_y^2(x^n), \ldots, \delta_y^{2n}(x^n)\right\},$$

where the last element is actually a scalar multiple of y^n. ∎

Let $h = z^2 - 4xy + 2z$ be the Casimir element of U. It is well known that h is a central element of U and that the center of U is the polynomial ring $K[h]$.

THEOREM 1.2. *$U_n = \oplus \sum_{i, j \geq 0, 2i+j \leq n} [h^i x^j]$ is a decomposition of U_n as a direct sum of irreducible U-submodules.*

Proof. First of all, h being central in U, it commutes with the action of U and therefore $[h^i x^j] \cong [x^j]$ as U-modules, hence $[h^i x^j]$ is irreducible as well. We prove first that the sum in the RHS is direct. Indeed, since

$$\sum_{2i+j\le n} [h^i x^j] = \sum_{j=0}^{n} \sum_{2i\le n-j} [h^i x^j] = \sum_{j=0}^{n} M_j$$

it suffices to show that $\sum_{j=0}^{n} M_j$ and $\sum_{2i\le n-j}[h^i x^j]$ are direct. For the first sum, we prove that $M_k \cap \sum_{j\ne k} M_j = 0$ for all k. If this is not the case, M_k and $\sum_{j\ne k} M_j$ have a common irreducible submodule N. But M_k is a sum of irreducible modules all isomorphic to $[x^k]$, so by the Jordan–Holder Theorem all the irreducible submodules of M_k are isomorphic to $[x^k]$. In particular $N \cong [x^k]$. On the other hand, $\sum_{j\ne k} M_j$ is a finite direct sum of irreducible submodules, each of which is isomorphic to an $[x^m]$ for some $m \ne k$. Again, the use of the Jordan–Holder Theorem makes $N \cong [x^m]$ for some $m \ne k$, a contradiction.

To see that $M_j = \sum_{2i\le n-j}[h^i x^j]$ is a direct sum, we will show that for all $s \ge 0$ the sum $\sum_{i=0}^{s}[h^i x^j]$ is direct. Argue this by induction on s. If $s = 0$, there is nothing to prove. Assume that the property is true for $s-1$, that is, $\sum_{i=0}^{s-1}[h^i x^j] \subseteq U_{j+2s-2}$ is direct. As $[h^s x^j]$ is irreducible, U_{j+2s-2} is a U-module, and $h^s x^j \notin U_{j+2s-2}$, we have $[h^s x^j] \cap U_{j+2s-2} = 0$ and therefore $[h^s x^j] \cap \sum_{i=0}^{s-1}[h^i x^j] = 0$. This together with the inductive hypothesis makes the sum $\sum_{i=0}^{s}[h^i x^m]$ direct and finishes the proof that the RHS is a direct sum.

It remains to show that $U_n = \oplus\sum_{2i\le n-j}[h^i x^j]$. Indeed, since $U_n \supseteq \oplus\sum_{2i\le n-j}[h^i x^j]$ and U_n is completely reducible, by the Jordan–Holder Theorem, it suffices to show that U_n and $\oplus\sum_{2i\le n-j}[h^i x^j]$ have the same number of direct summands. The number of composition factors in U_n is $\dim_K V_0 + \dim_K V_1$, where V_0 and V_1 are respectively the 0- and 1-weight subspaces of U_n (see Humphreys [10, p. 33]). But in U_n all the weights are even and hence the desired number is

$$\begin{aligned}\dim_K V_0 &= \left|\left\{x^i y^k z^j \in U_n \,\middle|\, \delta_z\left(x^i y^k z^j\right) = 0\right\}\right|\\ &= \left|\left\{x^i y^i z^j \,\middle|\, 2i+j \le n\right\}\right|\\ &= \left|\left\{(i,j) \,\middle|\, 2i+j \le n\right\}\right|\end{aligned}$$

which clearly equals the number of direct summands in $\oplus\sum_{2i\le n-j}[h^i x^j]$. ∎

COROLLARY 1.3. *Let V be an irreducible submodule of U_n. Then*

(i) $V \cong [x^j]$ *for some* $j \le n$.

(ii) $V \cong [x^j]$ *if and only if* $V = [P(h)x^j]$ *for some polynomial* $P(h) \ne 0$.

(iii) *If* $f = \sum_{i=0}^{s} P_i(h)x^{n_i}$, *where* $P_i \neq 0$ *and the* n_i *are distinct, then*

$$[f] = \oplus \sum_{i=0}^{s} [P_i(h)x^{n_i}] \cong \oplus \sum_{i=0}^{s} [x^{n_i}].$$

Proof. (i) This follows easily from the Jordan–Holder Theorem.

(ii) Assume $V \cong [x^j]$. Since

$$V \subseteq U_n = \oplus \sum_{i,m} [h^i x^m]$$

it follows that V can have a nonzero projection to $[h^i x^m]$ only when $m = j$. Thus

$$V \subseteq \oplus \sum_i [h^i x^j].$$

On the other hand $V \cong [x^j]$ is generated by its highest weight vector of weight $2j$. But the vectors of weight $2j$ in $\sum_i [h^i x^j]$ are all of the form $\sum_i a_i h^i x^j = P(h)x^j$ and the result follows.

(iii) Clearly

$$[f] \subseteq \sum_{i=0}^{s} [P_i(h)x^{n_i}].$$

Note that $[P_i(h)x^{n_i}] \cong [x^{n_i}]$ and therefore the above sum is direct since it is a sum of nonisomorphic irreducible modules. Also $[f]$ projects nontrivially to each summand $[P_i(h)x^{n_i}]$, so $[x^{n_i}]$ is a composition factor for $[f]$. Thus $\dim[f] \geq \sum \dim[x^{n_i}] = \dim \sum [P_i(h)x^{n_i}]$ and we get equality. ∎

The following theorem gives a family of generators for all the two-sided ideals of U. This family consists of highest weight vectors of the finite dimensional irreducible submodules of U.

THEOREM 1.4. *Let I be a two-sided ideal of U. Then there exist nonnegative integers $k, n_1, n_2, \ldots, n_k$ with $k \neq 0$ and polynomials $P_1, P_2, \ldots, P_k \in K[T]$ so that*:

$$I = \left(x^{n_1}P_1(h), x^{n_2}P_2(h), \ldots, x^{n_k}P_k(h)\right). \tag{2}$$

Proof. Note that any two-sided ideal of U is a U-module under the adjoint Lie action. U being a Noetherian ring, all its two-sided ideals are finitely generated as (one-sided, hence as) two-sided ideals and therefore it suffices to prove the case $I = (f)$, where $f \in U$. Say $f \in U_n$ for some n. Then $[f]$ is a direct sum of irreducibles so by Corollary 1.4(i), (ii)

$$(f) \supseteq [f] = \sum_{i=1}^{k} [P_i(h)x^{n_i}],$$

where n_i are not necessarily distinct. Thus $P_i(h)x^{n_i} \in (f)$. Conversely,

$$f \in [f] = \sum_{i=1}^{k} [P_i(h)x^{n_i}] \subseteq (P_1(h)x^{n_1}, \dots, P_k(h)x^{n_k})$$

so $(f) = (P_1(h)x^{n_1}, \dots, P_k(h)x^{n_k})$. ∎

LEMMA 1.5. *If $n \geq 0$ and $P, q \in K[T]$, then*

$$(x^n P(h), x^n Q(h)) = (x^n \operatorname{gcd}(P(h), Q(h)).$$

Proof. Have

$$\begin{aligned}
&(x^n P(h), x^n Q(h)) \\
&\quad = (x^n P(h)) + (x^n Q(h)) = (x^n)P(h) + (x^n)Q(h) \\
&\quad = (x^n)K[h]P(h) + (x^n)K[h]Q(h), \\
&\qquad\qquad\qquad\qquad \text{as } (x^n) \subseteq (x^n)K[h] \subseteq (x^n)U = (x^n) \\
&\quad = (x^n)\{K[h]P(h) + K[h]Q(h)\}, \qquad \text{where } \{\cdots\} \text{ is a set} \\
&\quad = (x^n)\{K[h]\operatorname{gcd}(P(h), Q(h))\}, \qquad \text{as } K[h] \text{ is a PID} \\
&\quad = (x^n)\operatorname{gcd}(P(h), Q(h)) = (x^n \operatorname{gcd}(P(h), Q(h))),
\end{aligned}$$

as desired. ∎

The following theorem gives a better set of generators for all the ideals of U and using this, we will deduce that all the ideals of U are just one-generated as two-sided ideals.

THEOREM 1.6. *Let I be an ideal of $U = U(sl_2(K))$. Then there exist integers $k > 0$ and $n_1 > n_2 > \cdots > n_k \geq 0$ and polynomials $P_1, P_2, \dots, P_k \in K[T]$ with $P_1 | P_2 | \cdots | P_k$ and $\deg(P_1) < \deg(P_2) < \cdots < \deg(P_k)$ such that*

$$I = (x^{n_1}P_1(h), x^{n_2}P_2(h), \dots, x^{n_k}P_k(h)). \tag{3}$$

Proof. Step 1. Start with $I = (x^{n_1}P_1(h), x^{n_2}P_2(h), \dots, x^{n_k}P_k(h))$ as in Theorem 1.4 and after rearranging the generators we may assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. If $n_i = n_{i+1}$ for some i, then by Lemma 1.5 we have

$$(x^{n_i}P_i(h), x^{n_i}P_{i+1}(h)) = (x^{n_i} \operatorname{gcd}(P_i(h), P_{i+1}(h))).$$

It follows that we can replace the two generators $x^{n_i}P_i(h)$ and $x^{n_i}P_{i+1}(h)$ of I by just one generator $x^{n_i} \operatorname{gcd}(P_i(h), P_{i+1}(h))$ and reduce the problem to $k - 1$ generators. Now k being finite, the process must stop in finitely many steps and therefore WLOG we may assume that $n_1 > n_2 > \cdots > n_k$.

Step 2. By the previous step may assume $n_1 > n_2 > \cdots > n_k$. Note that

$$\begin{aligned}
&\left(x^{n_{k-1}}P_{k-1}(h), x^{n_k}P_k(h)\right)\\
&\quad = \left(x^{n_{k-1}}P_{k-1}(h), x^{n_{k-1}}P_k(h), x^{n_k}P_k(h)\right), \qquad \text{since } n_{k-1} > n_k\\
&\quad = \left(x^{n_{k-1}}P_{k-1}(h), x^{n_{k-1}}P_k(h)\right) + \left(x^{n_k}P_k(h)\right)\\
&\quad = \left(x^{n_{k-1}}\gcd\left(P_{k-1}(h), P_k(h)\right)\right) + \left(x^{n_k}P_k(h)\right), \qquad \text{by Lemma 1.5}\\
&\quad = \left(x^{n_{k-1}}\gcd\left(P_{k-1}(h), P_k(h)\right), x^{n_k}P_k(h)\right).
\end{aligned}$$

Now replace P_{k-1} by $\gcd(P_{k-1}, P_k)$ as generators for I and hence WLOG we may assume $P_{k-1}|P_k$. Next, repeat this process with $k-1$ instead of k and obtain $P_{k-2}|P_{k-1}$, and so on. At the end we can write

$$I = \left(x^{n_1}P_1(h), x^{n_2}P_2(h), \dots, x^{n_k}P_k(h)\right),$$
$$\text{where } n_1 > n_2 > \cdots > n_k \text{ and } P_1|P_2|\cdots|P_k.$$

Step 3. We assume in the above expression that $\deg(P_i) = \deg(P_{i+1})$ for some i. As $P_i|P_{i+1}$, we can write $P_i = aP_{i+1}$ for some scalar $a \in K$, $a \neq 0$. It follows that

$$\begin{aligned}
\left(x^{n_i}P_i(h), x^{n_{i+1}}P_{i+1}(h)\right) &= \left(x^{n_i}aP_{i+1}(h), x^{n_{i+1}}P_{i+1}(h)\right)\\
&= \left(x^{n_i}P_{i+1}(h), x^{n_{i+1}}P_{i+1}(h)\right)\\
&= \left(x^{n_{i+1}}P_{i+1}(h)\right).
\end{aligned}$$

The moral is that, whenever $\deg(P_i) = \deg(P_{i+1})$, we can eliminate $x^{n_i}P_i(h)$ as a generator for I. But there are only finitely many generators, hence the process ends in finitely many steps. At the end we obtain a generating set for I satisfying the conclusion of the theorem. ∎

COROLLARY 1.7. *Every ideal I of U can be generated as a two-sided ideal by just one element.*

Proof. If $I = (0)$, the result holds. If $I \neq (0)$, by the Theorem 1.6, we can write

$$I = \left(x^{n_1}P_1(h), x^{n_2}P_2(h), \dots, x^{n_k}P_k(h)\right),$$

where $n_1 > n_2 > \cdots > n_k$, $P_1|P_2|\cdots|P_k$ and $\deg(P_1) < \deg(P_2) < \cdots < \deg(P_k)$. Consider

$$f = x^{n_1}P_1(h) + x^{n_2}P_2(h) + \cdots + x^{n_k}P_k(h).$$

By Corollary 1.3 we have

$$\begin{aligned}[f] &= [x^{n_1}P_1(h)] \oplus [x^{n_2}P_2(h)] \oplus \cdots \oplus [x^{n_k}P_k(h)] \\ &= [x^{n_1}P_1(h), x^{n_2}P_2(h), \dots, x^{n_k}P_k(h)]\end{aligned}$$

hence $(f) = I$.

Remark 1.8. The above result is not true for one-sided ideals. There is an example of a left ideal of $U = U(sl_2)$ that requires at least three generators (see Smith [15, Corollary 7.4]).

2. DISTURBING THE MODULE STRUCTURE: THE IDEAL (x^n)

The following computation contains the motivation for the whole section:

$$\begin{aligned}\delta_z(x^n) &= 2nx^n \qquad \text{for all } n \geq 0 \text{ is proved in Lemma 1.1,} \\ \delta_y(x^n) &= \delta_y(x)x^{n-1} + x\delta_y(x)x^{n-2} + \cdots + x^{n-2}\delta_y(x)x + x^{n-1}\delta_y(x) \\ &= -zx^{n-1} - xzx^{n-2} - \cdots - x^{n-2}zx - x^{n-1}z \\ &= -\left[x^{n-1}z + \delta_z(x^{n-1})\right] - x\left[x^{n-2}z + \delta_z(n^{n-2})\right] - \cdots - x^{n-1}z \\ &= -x^{n-1}z - 2(n-1)x^{n-1} - x^{n-1}z \\ &\quad\; - 2(n-2)x^{n-1} - \cdots - 2x^{n-1} \\ &= -n\left[x^{n-1}z + (n-1)x^{n-1}\right]\end{aligned}$$

hence

$$x^{n-1}z \equiv (1-n)x^{n-1} \qquad \operatorname{mod}(x^n)$$

and

$$x^{n-1}z^2 \equiv (1-n)^2 x^{n-1} \qquad \operatorname{mod}(x^n).$$

Thus

$$\begin{aligned}x^{n-1}h &= x^{n-1}(z^2 + 4xy - 2z) \\ &\equiv \left[(1-n)^2 - 2(1-n)\right]x^{n-1} \equiv (n^2-1)x^{n-1} \qquad \operatorname{mod}(x^n).\end{aligned}$$

An immediate consequence of this is the following:

PROPOSITION 2.1. *Consider the integer $n \geq 1$ and the polynomial $P \in K[T]$. Then*:

(i) $x^{n-1}(h - n^2 + 1) \in (x^n)$,

(ii) $x^{n-1}P(h) \equiv x^{n-1}P(n^2 - 1) \bmod (x^n)$,

(iii) *If* $0 \leq s < n$, *then* $(x^s \sum_{i=s+1}^{n} (h - i^2 + 1)) \subseteq (x^n)$.

Proof. First of all, (i) and (ii) are immediate from the previous computation. To prove (iii) argue by "downward" induction on s. This means that we induct on $n - s$. If $s = n - 1$, then the result is an easy consequence of (i). Assume now the property true for $s + 1$ and n, that is,

$$\left(x^{s+1}(h - n^2 + 1)\left(h - (n-1)^2 + 1\right) \cdots \left(h - (s+2)^2 + 1\right)\right) \subseteq (x^n).$$

Also, using (i) for the case $n = s + 1$ we get

$$\left(x^s\left(h - (s+1)^2 + 1\right)\right) \subseteq (x^{s+1}).$$

Multiplying this last relation by $(h - n^2 + 1)(h - (n-1)^2 + 1) \cdots (h - (s+2)^2 + 1)$, viewed either as a polynomial in h or as a product of ideals of U generated by linear polynomials in h, obtain

$$\begin{aligned}
&\left(x^s(h - n^2 + 1)\left(h - (n-1)^2 + 1\right) \cdots \left(h - (s+1)^2 + 1\right)\right) \\
&\quad \subseteq \left(x^{s+1}(h - n^2 + 1)\left(h - (n-1)^2 + 1\right) \cdots \left(h - (s+2)^2 + 1\right)\right) \\
&\quad \subseteq (x^n).
\end{aligned}$$

The last inclusion holds by the use of the inductive hypothesis, and we conclude that the property is true for s. This finishes the proof by induction. ∎

Now we continue our computation, this time in $[x^{n+1}]$.

$$\begin{aligned}
\delta_y(x^{n+1}) &= -(n+1)(x^n z + n x^n) \\
\delta_y^2(x^{n+1}) &= -(n+1)\,\delta_y(x^n z + n x^n) \\
&= -(n+1)\left[-n\left(x^{n-1}z + (n-1)x^{n-1}\right)z + 2x^n y \right. \\
&\qquad\qquad \left. + n(-n)\left(x^{n-1}z + (n-1)x^{n-1}\right)\right] \\
&= n(n+1)x^{n-1}z^2 + n(n+1)(2n-1)x^{n-1}z \\
&\qquad + n^2(n+1)(n-1)x^{n-1} - 2(n+1)x^n y.
\end{aligned}$$

Also

$$x^{n-1}(h - n^2 + 1) = x^{n-1}z^2 - 2x^{n-1}z - (n+1)(n-1)x^{n-1} + 4x^n y.$$

It follows that

$$\begin{aligned}
&\delta_y^2(x^{n+1}) - n(n+1)x^{n-1}(h - n^2 + 1) \\
&\quad = n(n+1)(2n+1)x^{n-1}z + n(n+1)(n-1)(2n+1)x^{n-1} \\
&\qquad - 2(n+1)(2n+1)x^n y \\
&\quad = (n+1)(2n+1)\big(-\delta_y(x^n) - n(n-1)x^{n-1}\big) \\
&\qquad + n(n+1)(n-1)(2n+1)x^{n-1} - 2(n+1)(2n+1)x^n y \\
&\quad = -(n+1)(2n+1)\delta_y(x^n) - 2(n+1)(2n+1)x^n y.
\end{aligned}$$

Hence, we have the relations

$$\begin{aligned}
x^n x &= x x^n = x^{n+1} \\
x^n z &= -\frac{1}{n+1}\delta_y(x^{n+1}) - n x^n \\
x^n y &= -\frac{1}{2(n+1)(2n+1)}\delta_y^2(x^{n+1}) - \frac{1}{2}\delta_y(x^n) \\
&\qquad + \frac{n}{2(2n+1)}x^{n-1}(h - n^2 + 1).
\end{aligned}$$

THEOREM 2.2. *$L[x^n]$ and $[x^n]L$ are both contained in*

$$[x^{n+1}] + [x^n] + [x^{n-1}](h - n^2 + 1).$$

Proof. Observe that $[x^n]L$ is a U-module. To see this, note that it is the span of the products of the form uv with $u \in [x^n]$ and $v \in L$ and on products the elements x, y, z act as derivations. Claim that $[x^n]L$ is generated by $x^n L$. Indeed, let V be the U-submodule generated by $x^n L$ and define $W \subseteq [x^n]$ by

$$W = \{t \in [x^n] \mid tL \subseteq V\}.$$

Then it is easy to see that W is a U-submodule of $[x^n]$ and since $x^n \in W$ we have $W = [x^n]$ and $V = [x^n]L$. Furthermore, since L is spanned by x, y, z we see that $[x^n]L$ is generated by $x^n x$, $x^n y$ and $x^n z$. By the computations, these three expressions are contained in the U-module

$$M = [x^{n+1}] + [x^n] + [x^{n-1}](h - n^2 + 1).$$

Thus the U-module they generate is also contained in M and hence $[x^n]L \subseteq M$. Finally, since $[x^n]$ is a U-module, it is clear that $L[x^n] \subseteq [x^n]L + [x^n] \subseteq M + [x^n] = M$. ∎

The next theorem gives the module structure for the ideal (x^n).

THEOREM 2.3. *Fix n and define the polynomials*

$$f_{ns}(h) = \begin{cases} 1 & \text{if } s \geq n \\ \prod_{i=s+1}^{n} (h - i^2 + 1) & \text{if } s < h. \end{cases}$$

Then

$$(x^n) = \oplus \sum_{s \geq 0} [x^s] f_{ns}(h) K[h]. \tag{4}$$

Proof. Let I denote the above right hand side. Note that $I \subseteq (x^n)$ since $x^s f_{ns}(h) \in (x^n)$ by Proposition 2.1. Observe also that all the terms in the expression of I are *isotypical components* (that is, sums of isomorphic irreducible modules of different type for different components) and since the direct sum is a finitary condition, the same Jordan−Holder argument used in the proof of the Theorem 1.2 yields the directness of the sum I. Conversely, since $x^n \in I$ (using $f_{nn}(h) = 1$), it suffices to show that $I \lhd U(L)$. For this, we need only to show that

$$L \cdot [x^s] f_{ns}(h) \subseteq I$$

and

$$[x^s] f_{ns}(h) \cdot L \subseteq I.$$

But this follows immediately from Theorem 2.2 applied to $[x^s]$, by the definition of f_{ns}. ∎

COROLLARY 2.4. $(x^n) \cap Z(U) = f_{n0}(h) K[h]$.

Proof. $(x^n) \cap Z(U)$ is the sum of the irreducible submodules of height zero in (x^n), that is, the isotypical component of height zero in (x^n). By Theorem 2.3, this is exactly $f_{n0}(h) K[h]$.

COROLLARY 2.5. *Set $x^0 = 1$ by convention.*

(i) *If $n \geq 0$, then $(x^{n+1}) < (x^n)$.*

(ii) *If $m, n \geq 0$, then $(x^m) = (x^n)$ if and only if $m = n$.*

Proof. (i) Clearly $(x^{n+1}) \subseteq (x^n)$ and since f_{n0} strictly divides $f_{n+1,0}$, by the previous corollary, we have

$$(x^{n+1}) \cap Z(U) = f_{n+1,0}(h) K[h] < f_{n0}(h) K[h] = (x^n) \cap Z(U).$$

It follows that $(x^{n+1}) \neq (x^n)$ and hence $(x^{n+1}) < (x^n)$. Part (ii) is immediate from (i). ▮

Note that the previous corollary can be proved easily using the fact that

$$x^n \in \operatorname{ann}_U V(n-1) \qquad \text{and} \qquad x^{n-1} \notin \operatorname{ann}_U V(n-1),$$

but Theorem 2.2 will be heavily used throughout this paper for other applications.

PROPOSITION 2.6. *If $n \geq 1$, then $(x^n, h - n^2 + 1) < U$.*

Proof. If $n = 1$, we obtain the case $(x) < U$ proved in Corollary 2.5, since $h \in (x)$. Assume $n \geq 2$ and suppose $(x^n, h - n^2 + 1) = U$. Then we have

$$\begin{aligned}
(x^{n-1}) &= (x^{n-1})U = (x^{n-1})(x^n, h - n^2 + 1) \\
&= (x^{n-1})\big((x^n) + (h - n^2 + 1)\big) \\
&\subseteq (x^{n-1})(x^n) + (x^{n-1})(h - n^2 + 1) \\
&\subseteq (x^n) + \big(x^{n-1}(h - n^2 + 1)\big) \qquad \text{as } h \text{ is central,} \\
&\subseteq (x^n) + (x^n) = (x^n), \qquad \text{by Corollary 2.1.}
\end{aligned}$$

This contradicts the previous result that $(x^n) < (x^{n-1})$. ▮

Remark 2.7. (i) Proposition 2.6 could have been proved better using the module structure given in Theorem 1.2 and Theorem 2.3. We can compute the isotypical component of height zero in $(x^n, h - n^2 + 1)$, see that it is not the whole center $Z(U)$, and the result holds. We chose to save this kind of argument for the more general context considered later.

(ii) For $n = 1$, Theorem 2.3 gives

$$(x) = \oplus \sum_{s \geq 0} [x^s] f_{1s}(h) K[h] = hK[h] \oplus \sum_{s \geq 1} [x^s] K[h] = \sum_{\substack{i, j \geq 0 \\ i + j \neq 0}} [x^i h^j].$$

We prove that $(x) = \oplus \sum_{i, j \geq 0 / i + j \neq 0} [x^i h^j]$ as follows. Let $V_{ij} \cong [x^i h^j]$ and map $V = \oplus \sum_{i, j} V_{ij}$ onto (x) in the obvious manner. We need to show that this is an isomorphism. If not, since V is completely reducible, the kernel must contain an irreducible submodule isomorphic to some V_{ij}. Say $W \cong V_{ij}$. Then a highest weight vector maps to 0. A highest weight vector in W is a sum

$$w = \sum_j k_j v_{ij},$$

where v_{ij} is a highest weight vector of V_{ij} and $v_{ij} \mapsto x^i h^j$. Thus $w \mapsto 0$ yields

$$x^i \sum_j k_j h^j = 0$$

a contradiction, since not all k_j are 0. Finally, $U(L) = (x) \oplus K$ since $(x) \supseteq [x] = L$, so we have the description of $U(L)$ derived in the previous section.

3. DOWN TO TWO GENERATORS: THE UNIQUENESS

Recall that by Corollary 1.7 every ideal I of U is generated as a two-sided ideal by just one element. Unfortunately, in the form given there, this element is not unique.

The next result says that if we start with an ideal I generated by two highest weight elements, then I can be generated by two highest weight elements, one of which being central of a particular form.

PROPOSITION 3.1. *If $0 \neq I \lhd U$ is as in Theorem 1.4 with $k \leq 2$ generators, then there exist an integer $n \geq 0$ and $P(h), Q(h) \in K[h]$ with $Q(h) | f_{n0}(h)$ and $h - n^2 + 1 | Q(h)$, whenever $n > 0$, so that*

$$I = (x^n P(h), P(h)Q(h)) = P(h)(x^n, Q(h)).$$

Proof. If $k = 1$, then $I = (x^n P(h)) = P(h)(x^n) = P(h)(x^n, f_{n0}(h))$ by Proposition 2.1(iii), hence I has the desired form with $Q(h) = f_{n0}(h)$.

Assume $k = 2$ and note that the statement of Theorem 1.6 can be reformulated as follows. Let I be an ideal of U and write $I = (x^{m_1} Q_1(h), \ldots, x^{m_k} Q_k(h))$ as in Theorem 1.4. Then there exist integers $0 < r \leq k$ and $n_1 > n_2 > \cdots > n_r > 0$ and polynomials $P_1 | P_2 | \cdots | P_r$ with $\deg P_1 < \deg P_2 < \cdots < \deg P_r$ so that $I = (x^{n_1} P_1(h), \ldots, x^{n_r} P_r(h))$. Applying this to our case $k = 2$, we can either go down to one generator, the case when we are done, or get

$$I = \big(x^n P_1(h), x^m P_1(h) P_2(h)\big) = P_1(h)\big(x^n, x^m P_2(h)\big),$$

where $n > m$ and $\deg P_2 > 0$. Claim that without loss of generality $P_2(h) | f_{nm}(h)$. Indeed, by Proposition 2.1 and Lemma 1.5 we have

$$\begin{aligned}\big(x^n, x^m P_2(h)\big) &= \big(x^n, x^m f_{nm}(h), x^m P_2(h)\big) \\ &= \big(x^n, x^m \gcd\big(f_{nm}(h), P_2(h)\big)\big).\end{aligned}$$

If $\gcd(f_{nm}(h), P_2(h)) = 1$, then $I = P_1(h)(x^m)$ and we are done by the case $k = 1$. Otherwise, change $P_2(h)$ to $\gcd(f_{nm}(h), P_2(h))$ and the claim is proved.

Claim now that $(x^n, x^m P_2(h)) = (x^n, P_2(h) f_{m0}(h))$. Indeed, by Proposition 2.1(iii), with $s = 0$, we have

$$P_2(h) f_{m0}(h) \in \big(x^m P_2(h)\big) \subseteq \big(x^n, x^m P_2(h)\big)$$

and one inclusion is clear. Conversely,

$$x^m f_{nm}(h) \in (x^n) \subseteq \left(x^n, P_2(h) f_{m0}(h)\right)$$

by Proposition 2.1(iii), with $s = m$,

and

$$x^m P_2(h) f_{m0}(h) \in \left(P_2(h) f_{m0}(h)\right) \subseteq \left(x^n, P_2(h) f_{m0}(h)\right)$$

imply that

$$x^m P_2(h) = x^m \gcd\left(f_{nm}(h), P_2(h) f_{m0}(h)\right) \in \left(x^n, P_2(h) f_{m0}(h)\right)$$

and hence $(x^n, x^m P_2(h)) \subseteq (x^n, P_2(h) f_{m0}(h))$.

Let $Q(h) = P_2(h) f_{m0}(h)$ and note that this divides $f_{n0}(h) = (h - n^2 + 1) \cdots (h - 1^2 + 1)$. Consider $r = \max\{s | h - s^2 + 1 | Q(h)\}$. Then $h - r^2 + 1 | Q(h) | f_{r0}(h)$ and claim that $(x^n, Q(h)) = (x^r, Q(h))$. Indeed, since $r \leq n$ we have $(x^r, Q(h)) \supseteq (x^n, Q(h))$. Conversely, from

$$x^r f_{nr}(h) \in (x^n) \subseteq (x^n, Q(h))$$

and

$$x^r Q(h) \in (Q(h)) \subseteq (x^n, Q(h))$$

we get

$$x^r = x^r \gcd\left(f_{nr}(h), Q(h)\right) \in (x^n, Q(h))$$

hence $(x^r, Q(h)) \leq (x^n, Q(h))$. Conclude that $I = P_1(h)(x^r, Q(h))$ with $h - r^2 + 1 | Q(h)$, $Q(h) | f_{r0}(h)$ and the proposition is proved. ∎

THEOREM 3.2. *Let I be a nonzero ideal of U. Then there exist an integer $n \geq 0$ and P, Q polynomials in h with $Q(h) | f_{n0}(h)$ and $h - n^2 + 1 | Q(h)$, whenever $n > 0$, so that*

$$I = (x^n P(h), P(h) Q(h)) = P(h)(x^n, Q(h)). \tag{5}$$

Proof. Write I as in the Theorem 1.4 with k generators and argue by induction on k. If $k < 2$, the result comes from Proposition 3.1. Let $k > 2$ and assume the theorem is true for $k - 1$. By Proposition 3.1 we have

$$\left(x^{n_1} P_1(h), x^{n_2} P_2(h)\right) = \left(x^a A(h), A(h) B(h)\right)$$

with $B(h) | f_{a0}(h)$ and $h - a^2 + 1 | B(h)$ if $a > 0$. Also, by the same result

$$\left(x^a A(h), x^{n_3} P_3(h)\right) = \left(x^c C(h), C(h) D(h)\right)$$

with a similar condition on $D(h)$. Putting these together, we obtain

$$\begin{aligned}
&(x^{n_1}P_1(h), x^{n_2}P_2(h), x^{n_3}P_3(h)) \\
&\quad = (x^a A(h), A(h)B(h), x^{n_3}P_3(h)) \\
&\quad = (x^c C(h), C(h)D(h), A(h)B(h)) \\
&\quad = (x^c C(h), \gcd(C(h)D(h), A(h)B(h)) \qquad \text{by Lemma 1.5.}
\end{aligned}$$

The moral is that we can change the first three generators for I into two and therefore I is also generated by $k-1$ highest weight elements. Using the inductive hypothesis the theorem is proved. ∎

Having the module structure for (x^n) we are now able to determine the module structure for $(x^n P(h), P(h)Q(h))$, that is, for an arbitrary ideal I of U.

THEOREM 3.3. *Let I be a nonzero ideal of U and write $I = (x^n P(h), P(h)Q(h))$ as in Theorem 3.2. Then*

$$I = \oplus \sum_{s \geq 0} [x^s] I_s, \tag{6}$$

where $I_s = P(h)\gcd(f_{ns}(h), Q(h))K[h]$.

Proof. Recall that by Theorem 2.3 we have

$$(x^n) = \oplus \sum_{s \geq 0} [x^s] f_{ns}(h) K[h].$$

Then

$$(x^n P(h)) = \oplus \sum_{s \geq 0} [x^s] P(h) f_{ns}(h) K[h].$$

Also,

$$\begin{aligned}
(P(h)Q(h)) &= P(h)Q(h)U = P(h)Q(h)\Big(\oplus \sum_{i,j \geq 0} [h^i x^j]\Big) \\
&= \oplus \sum_{s \geq 0} [x^s] P(h)Q(h)K[h].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(x^n P(h), P(h)Q(h)) \\
&\quad = \oplus \sum_{s \geq 0} [x^s] P(h) f_{ns}(h) K[h] + \oplus \sum_{s \geq 0} [x^s] P(h)Q(h)K[h] \\
&\quad = \oplus \sum_{s \geq 0} [x^s] (P(h) f_{ns}(h) K[h] + P(h)Q(h)K[h]) \\
&\quad = \oplus \sum_{s \geq 0} [x^s] P(h)\gcd(f_{ns}(h), Q(h))K[h]. \quad ∎
\end{aligned}$$

The next theorem says that the expression given in Theorem 3.3 is unique.

THEOREM 3.4. *Let I be a nonzero ideal of U and write*

$$I = \sum_{s \geq 0} [x^s] I_s = \sum_{s \geq 0} [x^s] J_s,$$

where $I_s, J_s \lhd K[h]$. Then $I_s = J_s$ for all $s \geq 0$.

Proof. Note that $[x^s]I_s = [x^s]J_s$, since they both represent the isotypical component type $[x^s]$ in I, that is, the sum of all irreducible submodules of I that are isomorphic to $[x^s]$. Taking the highest weight subspace in both sides, we have $x^s I_s = x^s J_s$, so $I_s = J_s$. ∎

COROLLARY 3.5. *If $I = (x^n P(h), P(h)Q(h))$ with $Q(h) | f_{n0}(h)$ and $h - n^2 + 1 | Q(h)$ whenever $n > 0$, then*

$$I \cap Z(U) = P(h)Q(h)K[h].$$

Proof. Note that $I \cap Z(U)$ is exactly the isotypical component of height zero in I and by Theorem 3.3 this is

$$I \cap Z(U) = P(h)\gcd(f_{n0}(h), Q(h))K[h] = P(h)Q(h)K[h]. \quad ∎$$

The following is a well-known elementary fact.

LEMMA 3.6. *If $A(h), B(h) \in K[h]$ are monic polynomials and $A(h)K[h] = B(h)K[h]$, then $A = B$.*

THEOREM 3.7. *Let I be a nonzero ideal of U. Write $I = (x^n P(h), P(h)Q(h))$ with P, Q monic, $Q(h) | f_{n0}(h)$, and $h - n^2 + 1 | Q(h)$, whenever $n > 0$, as in Theorem 3.2. Then n, P, Q satisfying the above properties are unique.*

Proof. Assume $I = (x^m P_1(h), P_1(h)Q_1(h))$ is another expression for I with the given properties. Then by Corollary 3.5

$$I \cap Z(U) = P(h)Q(h)K[h] = P_1(h)Q_1(h)K[h],$$

so $PQ = P_1 Q_1$ by Lemma 3.6. Furthermore, if $r \geq \max(n, m)$, then Corollary 3.5 yields

$$I_r = P(h)\gcd(f_{nr}(h), Q(h))K[h] = P_1(h)\gcd(f_{mr}(h), Q_1(h))K[h],$$

hence $I_r = P(h)K[h] = P_1(h)K[h]$ and as before we must have $P(h) = P_1(h)$. It follows that $Q(h) = Q_1(h)$. At this point

$$I = (x^n P(h), P(h)Q(h)) = (x^m P(h), P(h)Q(h)).$$

Finally, our hypothesis says that $n = 0$ iff $Q = 1$ iff $m = 0$. Also, in the case when both $n, m \neq 0$, by the same hypothesis $n^2 - 1$ and $m^2 - 1$ both represent the largest root of $Q(h)$, hence $n = m$. ∎

If $A(h), B(h) \in K[h]$, write $A(h) \lfloor B(h)$ instead of "$A(h)$ strictly divides $B(h)$." The next theorem is equivalent to Theorem 3.7. It actually gives all the presentations with minimal number of highest weight generators that an ideal can possibly have.

THEOREM 3.7′. *Let $0 \neq I \lhd U$. Then I is uniquely presented by generators in exactly one of the following forms*:

$$I = (x^n P(h)), \qquad \text{where } n \geq 0 \text{ and } P \text{ monic polynomial in } K[h] \tag{7}$$

or

$$I = (x^n P(h), P(h)Q(h)),$$
$$\text{where } n \geq 2,\ P, Q \in K[h] \text{ are monic and } h - n^2 + 1 \mid Q(h) \lfloor f_{n0}(h). \tag{8}$$

Proof. To prove the existence, we start with the expression given in Theorem 3.7 and assume that the conditions (8) are not all satisfied. Claim that I can be expressed as in (7). There are several cases to be considered. If $n = 0$, then $Q(h) | f_{00}(h) = 1$ implies $Q(h) = f_{00}(h)$. If $n = 1$, then $h - 1^2 + 1 | Q(h) | f_{10}(h) = h - 1^2 + 1$ implies $Q(h) = f_{10}(h)$. The only case we need to prove is when $Q(h) = f_{n0}(h)$. But in this case $I = (x^n P(h), P(h) f_{n0}(h)) = (x^n P(h))$ by Proposition 2.1(iii) with $s = 0$ and the fact that h is central. Thus I has the expression (7), as desired.

For uniqueness, we need to prove that I cannot have at the same time

(i) two expressions as in (8),

(ii) two expressions as in (7), or

(iii) one expression as in (7) and one as in (8).

Indeed, (i) comes clearly from Theorem 3.7. To prove (ii), we start with $I = (x^n P(h)) = (x^{n_1} P_1(h))$. Then by Proposition 2.1(iii) with $s = 0$, we have

$$I = \left(x^n P(h), P(h) f_{n0}(h)\right) = \left(x^{n_1} P_1(h), P_1(h) f_{n_1,0}(h)\right)$$

and the uniqueness in Theorem 3.7 gives $n = n_1$ and $P = P_1$. For (iii) we assume $I = (x^n P(h), P(h)Q(h)) = (x^{n_1} P_1(h))$ where $n \geq 2$, $h - n^2 + 1 | Q(h) \lfloor f_{n0}(h)$, and P, Q, P_1 are monic. As in (ii), write $(x^{n_1} P_1(h)) = (x^{n_1} P_1(h), P_1(h) f_{n_1,0}(h))$ and by Theorem 3.7 we must have $Q(h) = f_{n_1,0}(h)$, a contradiction. ∎

THEOREM 3.7″. *Let $0 \neq I \lhd U$. Then there exist a unique integer $n \geq 0$ and polynomials P, Q with $Q(h) | f_{n0}(h)$ and $h - n^2 + 1 | Q(h)$, whenever $n > 0$, so that*

$$I = \left(x^n P(h) + P(h) Q(h) \right).$$

Proof. This is clear from Theorem 3.7 and Corollary 1.7. ∎

Remark 3.8. If $I = (x^n P(h), P(h)Q(h))$ is as in Theorem 3.3, then

$$I_0 = P(h)Q(h)K[h] \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{n-1} < I_n = I_{n+1} = I_{n+2} = \cdots,$$

and thus n is the largest integer r so that $I_{r-1} < I_r$. This is another way to prove that n is unique in the previous theorem. If $n = 0$, the sequence $(I_s)_{s \geq 0}$ is constant. This corresponds to the case where the second generator for I is superfluous, that is, $I = (P(h))$.

Proof. Since $f_{nm}(h) | f_{n,m+1}(h)$ we have

$$\begin{aligned} I_m &= P(h)\gcd\left(f_{nm}(h), Q(h)\right)K[h] \\ &\subseteq P(h)\gcd\left(f_{n,m+1}(h), Q(h)\right) = I_{m+1}. \end{aligned}$$

Also, $f_{nm}(h) = 1$ for $m \geq n$, hence $I_m = P(h)\gcd(1, Q(h))K[h] = P(h)K[h]$. Finally, if $n > 1$, then

$$\begin{aligned} I_{n-1} &= P(h)\gcd\left(f_{n,n-1}(h), Q(h)\right)K[h] \\ &= P(h)\gcd\left(h - n^2 + 1, Q(h)\right)K[h] \\ &= P(h)(h - n^2 + 1)K[h] < P(h)K[h] = I_n. \quad ∎ \end{aligned}$$

The next three results deal with the sum, intersection, and inclusion between two ideals in terms of their generators.

PROPOSITION 3.9. *Let $I = (x^n P(h), P(h)Q(h))$, $I' = (x^{n'} P'(h), P'(h)Q'(h))$, and $I'' = (x^{n''} P''(h), P''(h)Q''(h))$ be three nonzero ideals of U written as in Theorem 3.7. If $I = I' + I''$, then $P(h) = \gcd(P'(h), P''(h))$, $P(h)Q(h) = \gcd(P'(h)Q'(h), P''(h)Q''(h))$, and n is either the largest integer r with $h - r^2 + 1 | Q(h)$ or $n = 0$ if no such r exists.*

Proof. Write $I' = \oplus\sum[x^s]I'_s$, $I'' = \oplus\sum[x^s]I''_s$ and $I = \oplus\sum[x^s]I_s$ as in Theorem 3.4. Then

$$I = I' + I'' = \sum_{s \geq 0} [x^s] I'_s + \sum_{s \geq 0} [x^s] I''_s = \sum_{s \geq 0} [x^s](I'_s + I''_s).$$

Now by Theorem 3.4 we must have $I_s = I'_s + I''_s$ for all s. In particular

$$\lim_{s \to \infty} I_s = \lim_{s \to \infty} I'_s + \lim_{s \to \infty} I''_s \qquad \text{and} \qquad I_0 = I'_0 + I''_0.$$

Now using Theorem 3.3 these mean

$$P(h)K[h] = P'(h)K[h] + P''(h)K[h] = \gcd(P'(h), P''(h))K[h]$$

and

$$\begin{aligned} P(h)Q(h)K[h] &= P'(h)Q'(h)K[h] + P''(h)Q''(h)K[h] \\ &= \gcd(P'(h)Q'(h), P''(h)Q''(h))K[h] \end{aligned}$$

and by Lemma 3.6 we must have $P = \gcd(P', P'')$ and $PQ = \gcd(P'Q', P''Q'')$. The assertion on n is clear from Theorem 3.7. ∎

PROPOSITION 3.10. *Let $I = (x^n P(h), P(h)Q(h))$, $I' = (x^{n'} P'(h), P'(h)Q'(h))$, and $I'' = (x^{n''} P''(h), P''(h)Q''(h))$ be three nonzero ideals of U written as in Theorem 3.7. If $I = I' \cap I''$, then $P = \operatorname{lcm}(P', P'')$, $PQ = \operatorname{lcm}(P'Q', P''Q'')$, and n is either the largest s with $h - s^2 + 1 | Q(h)$ or $n = 0$ if no such s exists.*

Proof. This is similar to the previous one. We only need to change "sum of ideals" by "intersection of ideals" and "gcd" by "lcm." ∎

PROPOSITION 3.11. *Consider $I = (x^n P(h), P(h)Q(h))$ and $I' = (x^{n'} P'(h), P'(h)Q'(h))$ as two ideals of U. Then $I \subseteq I'$ if and only if $P'|P$ and $P'Q'|PQ$.*

Proof. Note that $I \subseteq I'$ if and only if $I + I' = I'$. By Proposition 3.10, since n is determined by Q, the latter occurs if and only if

$$P' = \gcd(P, P') \Leftrightarrow P'|P$$

and

$$P'Q' = \gcd(PQ, P'Q') \Leftrightarrow P'Q'|PQ. \quad ∎$$

PROPOSITION 3.12. *The lattice of two-sided ideals of U is distributive.*

Proof. Let I, J_1, J_2 be three ideals of U. It suffices to show that

$$I + J_1 \cap J_2 = (I + J_1) \cap (I + J_2).$$

Indeed, if we write $I = (x^n P(h), P(h)Q(h))$ and $J_i = (x^{n_i} P_i(h), P_i(h)Q_i(h))$ with $i = 1, 2$ and the usual properties, then, using Propositions 3.9 and 3.10, the desired relation is equivalent to

$$\gcd(P, \operatorname{lcm}(P_1, P_2)) = \operatorname{lcm}(\gcd(P, P_1), \gcd(P, P_2))$$

and

$$\gcd(PQ, \operatorname{lcm}(P_1Q_1, P_2Q_2)) = \operatorname{lcm}(\gcd(PQ, P_1Q_1), \gcd(PQ, P_2Q_2)).$$

Note that the first relation is true since it is equivalent to the distributivity of the lattice of ideals in $K[h]$ which is a PID, and the second relation is a particular case of the first one. ∎

4. THE CLASSIFICATION OF THE PRIME IDEALS OF U

In this section we classify the prime ideals of $U = U(sl_2)$. We prove that the classical Krull dimension of U is 2. Specifically, each prime ideal of height one is generated by an (unique, by the Theorem 3.7) irreducible polynomial in the central indeterminate $h, h = z^2 + 4xy - 2z$ and each height two prime ideal of U has the form $(x^n, h - n^2 + 1)$, $n = 1, 2, 3, \cdots$.

To get started, note that since U is a domain, (0) is a prime ideal. Recall from Lemma 2.1 that (x) is a maximal (hence prime) ideal of U of codimension one. In order to determine all the prime ideals of U, we look at the expressions (7) and (8) and try to find conditions on the generators so that the corresponding ideal is prime. The following two results tell us when an ideal of the form (7) or (8) cannot be a prime ideal.

PROPOSITION 4.1. *Let* $0 \neq I \triangleleft U$, $I = (x^n P(h))$, *where* $n \geq 0$ *and* P *is monic polynomial as in* (7). *If either*:

(i) $\deg(P) = 0$ *and* $n \geq 2$ *or*

(ii) $n = 0$ *and* P *not an irreducible polynomial,*

then I *cannot be a prime ideal of* U.

Proof. Assume first that (i) holds, so $I = (x^n)$, $n \geq 2$. Let $A = (x^{n-1})$ and $B = (x^n, h - n^2 + 1)$. To prove that I is not prime it suffices to show that $AB \subseteq I$, $I < A$, and $I < B$. Indeed,

$$
\begin{aligned}
AB &= (x^{n-1})(x^n, h - n^2 + 1) = (x^{n-1})\big((x^n) + (h - n^2 + 1)\big) \\
&\subseteq (x^{n-1})(x^n) + (x^{n-1})(h - n^2 + 1) \subseteq (x^n) + \big(x^{n-1}(h - n^2 + 1)\big) \\
&\subseteq (x^n) = I,
\end{aligned}
$$

where the last inclusion holds by Proposition 2.1. Also $I < A$, by Corollary 2.5. Finally, it is clear that $I \subseteq B$ and if $I = B$, then the uniqueness of the generators in Theorem 3.7 is contradicted. Conclude that $I < B$.

Assume now that (ii) is true. Then $P = QR$ with Q, R monic and nonconstant. Then,

$$I = (P(h)) = (Q(h)R(h)) = (Q(h))(R(h)).$$

Clearly $I \subseteq (Q(h))(R(h))$ and since $Q|P$ and $R|P$ we have $I \subseteq (Q(h))$ and $I \subseteq (R(h))$. By the uniqueness of the generator coming from Theorem 3.7, the equality cannot hold in any of the previous inclusions. Therefore we have $I < (Q(h))$, $I < R(h))$, and $I \subseteq (Q(h))(R(h))$, which means that I is not a prime ideal. ∎

PROPOSITION 4.2. *Let* $0 \neq I \lhd U$ *with* $I = (x^n P(h), P(h)Q(h))$, $n \geq 2$, P, Q *monic and* $h - n^2 + 1 | Q(h) \lfloor f_{n0}(h)$. *If either*

(i) $P \neq 1$ *or*

(ii) $P = 1$ *and* $Q(h) \neq h - n^2 + 1$,

then I cannot be a prime ideal of U.

Proof. Assume (i) is true so $I = (x^n P(h), P(h)Q(h)) = (x^n, Q(h))(P(h))$. Moreover, $I \subseteq (P(h))$ as the generators for I are multiples of $P(h)$. Also, $I \subseteq (x^n, Q(h))$ since $x^n P(h) \in (x^n)$ and $P(h)Q(h) \in (Q(h))$. In any of these inclusions, by the uniqueness Theorem 3.7, the equality does not hold. Therefore $I < (P(h))$, $I < (x^n, Q(h))$, and $I \supseteq (P(h))(x^n, Q(h))$ which means that I is not prime.

Assume now that (ii) holds, that is, $I = (x^n, Q(h))$ with Q satisfying the divisibility condition (8). The hypothesis $Q(h) \neq h - n^2 + 1$ means that Q has degree at least two. Write $Q(h) = (h - n^2 + 1)Q_1(h)$, where

$$Q_1(h) = \left(h - n_1^2 + 1\right)\left(h - n_2^2 + 1\right) \cdots \left(h - n_t^2 + 1\right)$$

for $t \geq 1$ and $n > n_1 > n_2 > \cdots > n_t > 0$.

Observe first that $(x^n, Q_1(h)) = (x^{n_1}, Q_1(h))$. Indeed, using Proposition 2.1 we have

$$\begin{aligned}\left(x^n, Q_1(h)\right) &= \left(x^n, x^{n_1} f_{n,n_1}(h), x^{n_1} Q_1(h), Q_1(h)\right) \\ &= \left(x^n, x^{n_1} \gcd\left(f_{n,n_1}(h), Q_1(h)\right), Q_1(h)\right) \\ &= \left(x^n, x^{n_1}, Q_1(h)\right) = \left(x^{n_1}, Q_1(h)\right).\end{aligned}$$

Here we have used Lemma 1.5 and the fact that $\gcd(f_{n,n_1}(h), Q_1(h)) = 1$. Consider the ideals $A = (x^n, h - n^2 + 1)$ and $B = (x^{n_1}, Q_1(h))$ and note that

$$\begin{aligned}I = (x^n, Q(h)) &= \left(x^n, (h - n^2 + 1)Q_1(h)\right) \\ &\supseteq (x^n, h - n^2 + 1)\left(x^n, Q_1(h)\right) = AB.\end{aligned}$$

The inclusion holds because

$$\begin{aligned}&(x^n)(x^n) + (x^n)\left(Q_1(h)\right) + (h - n^2 + 1)(x^n) + (h - n^2 + 1)\left(Q_1(h)\right) \\ &\quad \subseteq (x^n) + (h - n^2 + 1)\left(Q_1(h)\right).\end{aligned}$$

There are two cases. First, if $Q_1(h)$ is a proper divisor of $f_{n_1,0}$, then $B = (x^{n_1}, Q_1(h))$ satisfies (8). Note that A also satisfies (8) and observe that $I \subseteq A$ and $I \subseteq B$ since $h - n^2 + 1$ and $Q_1(h)$ both divide $Q(h)$. The equality cannot hold by the uniqueness of the generators in the form (8) given by Theorem 3.7.

The second case is when $Q_1(h) = f_{n_1,1}$. In this case $Q_1(h) \in (x^{n_1})$ by the Proposition 2.1(iii), hence $B = (x^{n_1}, Q_1(h)) = (x^{n_1})$, the last expression being as in (7). As before, we have $I \subseteq A$ and $I \subseteq B$, since $x^n \in (x^{n_1})$ and $Q(h) \in (Q_1(h)) \subseteq (x^{n_1})$. Another use of the uniqueness Theorem 3.7 gives $I < A$ and $I < B$. To summarize, we obtained $A, B \triangleleft U$ with $I < A$, $I < B$, and $I \supseteq AB$. This means that I cannot be a prime ideal of U. ∎

Having Propositions 4.1 and 4.2 in hand and analyzing what we have left from the whole list of ideals given in (7) and (8), we can say that the nonzero ideals of U that are candidates for being prime ideals are of the following form:

(i) (x), or

(ii) $(P(h))$, where P is an irreducible polynomial, or

(iii) $(x^n, h - n^2 + 1)$, where $n \geq 2$.

The case (i) is clear, (x) being the augmentation ideal of U. The next two propositions prove that any ideal of U of the form (ii) or (iii) is actually a prime ideal. This will enable us to give the complete list of prime ideals of U and to classify them by height.

PROPOSITION 4.3. *If $n \geq 2$, then the ideal $I = (x^n, h - n^2 + 1)$ is maximal, hence prime.*

Proof. Assume $I \subseteq J < U$ and write $J = (x^m P(h), P(h)Q(h))$ with the obvious conditions. Now by Proposition 3.11 we must have $P(h)|1$ and $P(h)Q(h)|h - n^2 + 1$. It follows that $P = 1$ and $Q(h)|h - n^2 + 1$. If $Q = 1$, then $J = U$. If $Q(h) = h - n^2 + 1$, then $m^2 - 1 = n^2 - 1$ is the largest root of Q, hence $m = n$ and $J = I$. This proves the maximality of I.

PROPOSITION 4.4. *If $P \in K[T]$ is a monic nonconstant irreducible polynomial, then the ideal $I = (P(h))$ is a prime ideal of U.* ∎

Proof. Suppose I is not prime. Then there exists ideals $A \not\subseteq I$ and $B \not\subseteq I$ so that $AB \subseteq I$. Claim first that WLOG we may assume that both A and B are as in (7). Indeed, if A is as in (8), then write

$$A = \left(x^n P_1(h), P_1(h)Q_1(h)\right) = \left(x^n P_1(h)\right) + \left(P_1(h)Q_1(h)\right) = A_1 + A_2$$

and observe that A_1 and A_2 are as in (7). Now $A \not\subseteq I$ implies either $A_1 \not\subseteq I$ or $A_2 \not\subseteq I$, say $A_1 \not\subseteq I$. Using this and $A_1 B \subseteq AB \subseteq I$, we can replace A by A_1. Therefore, WLOG we may assume that A and (with a similar reasoning) B are as in (7).

Consider now

$$A = (x^n Q(h)) \qquad \text{and} \qquad B = (x^m R(h)),$$

where $m, n \geq 0$ and Q, R are monic polynomials. The hypotheses $A \not\subseteq I$ and $B \not\subseteq I$ tell us that $x^n Q(h) \notin (P(h))$ and $x^m R(h) \notin (P(h))$. On the other hand $x^n Q(h) x^m R(h) = x^{n+m} Q(h) R(h) \in AB \subseteq I = (P(h))$ and $x^{n+m} P(h) \in I$. Therefore

$$x^{n+m} \gcd(Q(h)R(h), P(h)) \in I$$

and as P is irreducible, there are two cases:

If $\gcd(QR, P) = 1$, then $x^{n+m} \in I = (P(h))$. It follows that $(x^{n+m}) \subseteq (P(h))$ and Proposition 3.11 gives $P|1$, a contradiction.

If $\gcd(QR, P) = P$, then $P(h)|Q(h)R(h)$ and since $P(h)$ is a prime element in the PID $K[h]$, we must have either $P(h)|Q(h)$ or $P(h)|R(h)$. Assume $P(h)|Q(h)$. Then $x^n Q(h) \in (P(h)) = I$ and hence $A = (x^n Q(h)) \subseteq (P(h)) = I$. This contradicts the hypothesis that $A \not\subseteq I$. The case $P(h)|R(h)$ is treated in a similar way. ∎

Recall that *the classical Krull dimension* of a ring R is the maximal length of an increasing chain of prime ideals of R. We are now able to formulate the Classification Theorem for the prime ideals of U.

THEOREM 4.5. *The enveloping algebra $U = U(sl_2)$ has classical Krull dimension* 2. *Precisely,*

(i) $\{0\} \cup \{(h - n^2 + 1) | n \geq 1\}$ *is the set of nonmaximal prime ideals of U,*

(ii) $\{(x^n, h - n^2 + 1) | n \geq 1\}$ *is the set of maximal ideals of height two in U, and*

(iii) $\{(P(h)) | P$ *is nonconstant irreducible monic,* $P(h) \neq h - n^2 + 1$, *for all* $n \geq 1\}$ *is the set of the height one maximal ideals in U.*

Proof. Note first that (x) is written in the nonstandard form (x, h). So far we have proved that the union of these three sets is the set of all prime ideals of U and by the uniqueness theorem these sets do not overlap and consist of distinct ideals. We also know that the ideals in (ii) are maximal.

Observe that $(h - n^2 + 1) \subseteq (x^n, h - n^2 + 1)$ for all $n \geq 2$ and $(h) \subseteq (x)$. The uniqueness theorem makes these inclusions strict. At this point we prove the theorem if we can show that all the prime ideals coming from the center have height one and that the ideal $(h - n^2 + 1)$ is the unique prime ideal of U properly included in $(x^n, h - n^2 + 1)$ for all $n \geq 1$.

Let $(Q(h))$ be an arbitrary prime ideal of the form (i) or (iii), where Q is monic irreducible. Clearly $(Q(h))$ has height at least one. To prove that it is exactly one, we need to show that it cannot contain any proper prime ideal. Observe that the inclusion $(x^n, h - n^2 + 1) \subseteq (Q(h))$ is impossible by the uniqueness theorem and the fact that $(x^n, h - n^2 + 1)$ is maximal for all $n \geq 1$. Therefore, the only possibility is when $(P(h)) \subseteq (Q(h))$ for some monic irreducible P. But in this case the Proposition 3.11 gives $Q(h)|P(h)$ and since P and Q are nonconstant monic irreducible, we conclude that $P = Q$.

Finally, note that the only candidates for proper prime subideals of $(x^n, h - n^2 + 1)$ are the ones coming from the center. Assume then $(P(h)) = (P(h)x^0, P(h)) \subseteq (x^n, h - n^2 + 1)$. By Proposition 3.11 we must have $h - n^2 + 1|P(h)$, hence $h - n^2 + 1 = P(h)$, as desired. ∎

Remark 4.6. It is well known (see McConnell and Robson [12, Corollary 9.1.8] or Passman [14, Theorem 20.6]) that the universal enveloping algebra $U(L)$ of a finite dimensional Lie algebra L is a *Jacobson ring*, that is, a ring where every prime ideal is an intersection of primitive ideals. Also recall that the maximal ideals are primitive and the primitive ideals are prime. If we restrict all of these to our context $U = U(sl_2)$ and consider $N_n = (h - n^2 + 1)$, $n \geq 1$, to be an arbitrary height one prime ideal in U then, by the proof of Theorem 4.5, the only prime ideals of U containing N_n are N_n and $M_n = (x^n, h - n^2 + 1)$. But N_n is an intersection of primitive ideals, hence it has to be itself a primitive ideal. As all the other nonzero primes of U are maximal, we conclude that all the nonzero prime ideals of U are primitive.

We should mention here a theorem of Dixmier that gives equivalent conditions for a prime ideal to be primitive (see [6; 7; 9, 8.5.7]): Let L be a complex semi-simple Lie algebra. Let J be a prime ideal of $U = U(L)$. The following conditions are equivalent:

(i) J is primitive;

(ii) $\dim Z(U)/J \cap Z(U) = 1$;

(iii) the center of U/J is C;

(iv) the intersection of the prime ideals of U strictly containing J is distinct from J.

5. PRODUCTS AND INTERSECTIONS OF PRIMES

Since we now know the structure of the prime ideals of U, an interesting question would be to characterize the ideals of U that are intersections of primes (called *radical ideals*) and the ideals of U that are products of primes.

As we have seen in the previous section, most of the prime ideal structure of U comes from its center $Z = K[h]$. Since there are ideals of the center which are not an intersection of prime ideals, we do not expect all the ideals of U to be intersections of primes. Moreover, we shall see in this section that the obstruction for the ideals of U to be intersections of primes actually comes from the center. The main result in this section is that every ideal of U is uniquely a product of prime ideals. For an ideal I of U presented as in (7) or (8) we will be able to determine the prime ideal factors and the radical by just looking at the formulas for the generators. At the end of this section we determine the generators for the sum and the intersection of two ideals, respectively, the *greatest common divisor* and the *least common multiple* formulas. We also study the relation between the expressions for the generators which is equivalent to saying that two ideals are included one in another.

Throughout this section let $M_n = (x^n, h - n^2 + 1)$ for $n \geq 1$, where $M_1 = (x) = (x, h - 1^2 + 1)$. Also $\sqrt{I}$ is the *radical* of I and it represents the intersection of all the prime ideals of U containing I.

PROPOSITION 5.1. *If $r > 0$ and $0 < n_1 < n_2 < \cdots < n_r$ are integers, then*

$$M_{n_1} \cap M_{n_2} \cap \cdots \cap M_{n_r} = \left(x^{n_r}, \prod_{i=1}^{r} \left(h - n_i^2 + 1 \right) \right).$$

Proof. Write $M_i = (x^{n_i} P_i(h), P_i(h) Q_i(h))$, where $P_i(h) = 1, Q_i(h) = h - n_i^2 + 1$ for $1 \leq i \leq r$. Also $M = \bigcap_{i=1}^{r} M_i = (x^n P(h), P(h) Q(h))$ is as in Theorem 3.7. Then by Proposition 3.10 we have $P = \operatorname{lcm}(P_1, P_2, \dots, P_r) = 1$ and $Q = PQ = \operatorname{lcm}(P_1 Q_1, P_2 Q_2, \dots, P_r Q_r) = \prod_{i=1}^{r} (h - n_i^2 + 1)$. Moreover, $n = \max\{s | h - s^2 + 1 | Q(h)\} = n_r$. ∎

Observe that if $r < n_r$ in the preceding result, then $(h - n_r^2 + 1) \times (h - n_{r-1}^2 + 1) \cdots (h - n_1^2 + 1)$ is a proper divisor of $f_{n_r, 0}$, hence $(x^{n_r}, (h - n_r^2 + 1)(h - n_{r-1}^2 + 1) \cdots (h - n_1^2 + 1))$ satisfies (8).

On the other hand if $r = n_r$, then $i = n_i$ for all $1 \leq i \leq r$ and $M_i \cap M_2 \cap \cdots \cap M_r = (x^r, (h - r^2 + 1)(h - (r-1)^2 + 1) \cdots (h - 1^2 + 1)) = (x^r)$ by Proposition 2.1. This last expression is as in (7).

PROPOSITION 5.2. *Let $n_1, n_2, \dots, n_r$ be distinct positive integers and P be a polynomial with $P(n_i^2 - 1) \neq 0$ for all $1 \leq i \leq r$. Then*

$$(P(h))\left(M_{n_1} \cap M_{n_2} \cap \cdots \cap M_{n_r}\right) = (P(h)) \cap M_{n_1} \cap M_{n_2} \cap \cdots \cap M_{n_r}.$$

Proof. Write $(P(h)) = (P(h), P(h)1)$. Using Proposition 5.1, the result follows immediately from Proposition 3.10 since

$$\operatorname{lcm}\left(P(h), \prod_{i=1}^{r} \left(h - n_i^2 + 1 \right) \right) = P(h) \prod_{i=1}^{r} \left(h - n_i^2 + 1 \right)$$

and

$$\operatorname{lcm}(P(h), 1) = P(h). \quad \blacksquare$$

Let I be a proper ideal of U. Then by the Theorem 3.7 we can write I uniquely in the form $I = (x^n P(h), P(h)Q(h))$ satisfying the conditions (5)

LEMMA 5.3. *With the above hypothesis, if $x^m \in I$ for some $m \geq 0$, then $P(h) = 1$.*

Proof. From $(x^m) = (x^m, f_{m0}(h)) \subseteq I = (x^n P(h), P(h)Q(h))$, by Proposition 3.11, we must have $P(h)|1$. $\blacksquare$

PROPOSITION 5.4. *If A is a finite set (with multiplicities) of positive integers, then*

$$\prod_{i \in A} M_i = \bigcap_{i \in A} M_i.$$

Proof. Observe first that $\prod_{i \in A} M_i \subseteq \bigcap_{i \in A} M_i$, since a product of ideals is included in their intersection. Next note that $x^i \in M_i$ for all $i \in A$ gives

$$x^{\sum_{i \in A} i} \in \prod_{i \in A} M_i.$$

By the use of the previous lemma, this forces the ideal $\prod_{i \in A} M_i$ to have a unique expression (5) with $P(h) = 1$. Now by Proposition 5.2 and the comments following it, any ideal of this form is a finite intersection of ideals of the form M_n. Write

$$\prod_{i \in A} M_i = \bigcap_{j \in B} M_j,$$

for some finite set B of positive integers. Moreover, if $j \in B$, then

$$M_j \supseteq \bigcap_{j \in B} M_j = \prod_{i \in A} M_i.$$

But M_j is prime, hence $M_j \supseteq M_i$ for some $i \in A$. The maximality of M_i forces $M_j = M_i$ and the uniqueness of the expression (8) gives $j = i \in A$. We deduce from here that $B \subseteq A$ and therefore

$$\prod_{i \in A} M_i = \bigcap_{j \in B} M_j \supseteq \bigcap_{i \in A} M_i.$$

This, together with the reverse inclusion proved at the beginning, gives

$$\prod_{i \in A} M_i = \bigcap_{i \in A} M_i,$$

as desired. $\blacksquare$

Remark 5.5. (1) $M_n^2 = M_n \cap M_n = M_n$.

(2) If $m, n > 0$, then $M_m M_n = M_m \cap M_n = M_n \cap M_m = M_n M_m$, hence any finite product of ideals of this kind does not depend on the order of the factors. In particular, $\prod_{i \in A} M_i$ is well defined. But any other prime ideal is generated by a central element, therefore any (finite) product of prime ideals of U does not depend on the order of the factors. We shall see in the next theorem that each ideal of U is uniquely a product of primes, so by the previous comments, any product of two ideals in U is commutative.

(3) Since $(x^n) = (x^n, f_{no}(h)) = \bigcap_{i=1}^n M_i = \prod_{i=1}^n M_i$, by the above observations we deduce that $(x^n)^2 = (x^n)$.

THEOREM 5.6. *If I is a nonzero proper ideal of U, then I can be written as a product of prime ideals. Assuming that all M_n factors occur to the first power, the expression is unique up to a permutation of factors.*

Proof. Write $I = (x^n P(h), P(h)Q(h)) = (P(h))(x^n, Q(h))$ as in Theorem 3.7 and let $P = P_1 P_2 \cdots P_t$ be the expression of P as a product of monic prime factors. If $n = 0$, then clearly $I = (P(h)) = (P_1(h))(P_2(h)) \cdots (P_t(h))$. If $n > 0$, then $Q(h) = \prod_{i=1}^r (h - n_i^2 + 1)$, where $0 < n_1 < n_2 < \cdots < n_r = n$, hence $(x^n, Q(h)) = \bigcap_{i=1}^r M_{n_i} = \prod_{i=1}^r M_{n_i}$ by Propositions 5.1 and 5.4. It follows that $I = (P_1(h))(P_2(h)) \cdots (P_t(h)) M_{n_1} M_{n_2} \cdots M_{n_r}$ is a prime ideal factorization for I and the existence is proved.

To prove the uniqueness, assume that

$$\begin{aligned} I &= (P_1(h))(P_2(h)) \cdots (P_t(h)) M_{n_1} M_{n_2} \cdots M_{n_r} \\ &= (Q_1(h))(Q_2(h)) \cdots (Q_s(h)) M_{m_1} M_{m_2} \cdots M_{m_q}, \end{aligned}$$

where P_i and Q_j are nonconstant monic irreducible for all $1 \le i \le t$ and $q \le j \le s$ and $0 < n_1 < n_2 < \cdots < n_r$, $0 < m_1 < m_2 < \cdots < m_q$. Note that we also allow q, r, s, t to be zero. Now from Propositions 5.2 and 5.4 we have

$$\begin{aligned} I &= P_1(h) P_2(h) \cdots P_t(h)(x^n, P(h)) \\ &= Q_1(h) Q_2(h) \cdots Q_s(h)(x^m, Q(h)), \end{aligned}$$

where $n = n_r$ and $P(h) = (h - n_1^2 + 1)(h - n_2^2 + 1) \cdots (h - n_r^2 + 1)$ if $r > 0, n = 0$, and $P(h) = 1$ and if $r = 0$, $m = m_q$, and $Q(h) = (h - m_1^2 + 1)(h - m_2^2 + 1) \cdots (h - m_q^2 + 1)$ if $q > 0$, $m = 0$, and $Q(h) = 1$ if $q = 0$. Now by Theorem 3.7 we must have $P_1 P_2 \cdots P_t = Q_1 Q_2 \cdots Q_s$, $n = m$, and $P = Q$. Moreover, using the fact that $K[h]$ is a UFD, we get $t = s$, $r = q$, and, up to a permutation of factors, $P_i = Q_i$ for $1 \le i \le t = s$ and $n_j = m_j$ for all $1 \le j \le r = q$. ∎

In the last part of this section we deal with the lattice of ideals of U. We shall study the inclusion, intersection, and the sum of two ideals in terms of their prime ideal factorizations. To do this, it will be useful to distinguish between the three classes of prime ideals defined in Theorem 4.5. Precisely, we can write an ideal I of U as a product of powers of distinct prime ideals of U as

$$I = I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_{n_1}^{b_1} N_{n_2}^{b_2} \cdots N_{n_s}^{b_s} M_{m_1} M_{m_2} \cdots M_{m_r},$$

where $M_m = (x^m, h - m^2 + 1)$ is a maximal ideal of height two, $N_n = (h - n^2 + 1)$ is a prime nonmaximal ideal, I_i is a maximal height one, and $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_s > 0$. Sometimes it is more useful to consider expressions of the form

$$I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_1^{b_1} N_2^{b_2} \cdots N_n^{b_n} M_1^{c_1} M_2^{c_2} \cdots M_n^{c_n},$$

where the exponents are allowed to be zero.

Let us change the topic a little bit and ask: How many prime ideals can contain a given ideal I? Well, there are only finitely many of them. To see this (using exclusively the theory developed in this paper) take $J \supseteq I$ to be a prime ideal. As I is uniquely a finite product of primes, J has to contain one of the prime factors of I. But there are only finitely many of them and, as we have seen before, each one is contained in at most two distinct prime ideals. It follows that there are only finitely many choices for J.

Recall that the radical ideals of a ring are the ones obtained as an intersection of prime ideals. In our case the radical ideals are finite intersections of primes. Also the radical of an ideal is the intersection of all the prime ideals containing it. In the next proposition we determine the factorization into prime ideals for all the radical ideals. We will prove that the intersection of a finite set of primes is equal to the product of the minimal members in the set.

LEMMA 5.7. *Let $I_1, I_2, \ldots, I_t$ be distinct maximal ideals height one in U, $N_{n_1}, N_{n_2}, \ldots, N_{n_r}, N_{n_{r+1}}, \ldots, N_{n_s}$ distinct nonmaximal prime ideals and $M_{n_1}, M_{n_2}, \ldots, M_{n_r}, M_{n'_{r+1}}, M_{n'_{r+2}}, \ldots, M_{n'_q}$ distinct maximal ideals height two, where $\{n_{r+1}, n_{r+2}, \ldots, n_s\} \cap \{n'_{r+1}, n'_{r+2}, \ldots, n'_q\} = \varnothing$. Then*

$$\begin{aligned} &I_1 \cap I_2 \cap \cdots \cap I_t \cap N_{n_1} \cap N_{n_2} \cap \cdots N_{n_r} \cap N_{n_{r+1}} \cap \cdots \cap N_{n_s} \cap M_{n_1} \\ &\qquad \cap M_{n_2} \cap \cdots \cap M_{n_r} \cap M_{n'_{r+1}} \cap \cdots \cap M_{n'_q} \\ &\quad = I_1 I_2 \cdots I_t N_{n_1} N_{n_2} \cdots N_{n_r} N_{n_{r+1}} \cdots N_{n_s} M_{n'_{r+1}} M_{n'_{r+2}} \cdots M_{r'_q}. \end{aligned}$$

Proof. We have

$$\begin{aligned} I_1 \cap I_2 \cap \cdots \cap I_t \cap N_{n_1} \cap N_{n_2} \cap \cdots \cap N_{n_r} \cap N_{n_{r+1}} \cap \cdots \cap N_{n_s} \\ \cap M_{n_1} \cap \cdots \cap M_{n_r} \cap M_{n'_{r+1}} \cap \cdots M_{n'_q} \\ = \left(I_1 \cap I_2 \cap \cdots \cap I_t \cap N_{n_1} \cap N_{n_2} \cap \cdots \cap N_{n_s}\right) \\ \cap \left(M_{n'_{r+1}} \cap \cdots \cap M_{n'_q}\right) \qquad \text{as } N_{n_i} \subseteq M_{n_i} \text{ for } 1 \leq i \leq r \\ = \left(I_1 I_2 \cdots I_t N_{n_1} N_{n_2} \cdots N_{n_s}\right) \cap \left(M_{n'_{r+1}} M_{n'_{r+2}} \cdots M_{n'_q}\right) \\ = I_1 I_2 \cdots I_t N_{n_1} N_{n_2} \cdots N_{n_s} M_{n'_{r+1}} M_{n'_{r+2}} \cdots M_{n'_q}. \end{aligned}$$

Here we have used Propositions 5.1, 5.4, and the facts that in $Z = K[h]$ the product of distinct prime ideals equals their intersection and in U the generator for the product of ideals coming from the center is the product of their generators. In particular, this generator does not vanish at $(n'_i)^2 - 1$ for all $r + 1 \leq i \leq q$ and Lemma 5.1 can be safely applied. ∎

Let I be an arbitrary proper ideal of U. By Theorem 5.6 and the previous notation, I can be uniquely expressed in the form

$$I = I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_{n_1}^{b_1} N_{n_2}^{b_2} \cdots N_{n_r}^{b_r} N_{n_{r+1}}^{b_{r+1}} \cdots N_{n_s}^{b_s} M_{n_1} M_{n_2} \cdots M_{n_r} M_{n'_{r+1}} \cdots M_{n'_q},$$

where $a_i > 0$, $b_j > 0$ for $1 \leq i \leq t$, $1 \leq j \leq s$ and

$$\{n_{r+1}, n_{r+2}, \ldots, n_s\} \cap \{n'_{r+1}, n'_{r+2}, \ldots, n'_q\} = \varnothing.$$

COROLLARY 5.8. *In the preceding context we have*

$$\sqrt{I} = I_1 I_2 \cdots I_t N_{n_1} N_{n_2} \cdots N_{n_s} M_{n'_{r+1}} \cdots M_{n'_q}.$$

Proof. Note that any prime ideal J containing I has to contain one of the prime factors of I. Therefore J lies in the set

$$\mathscr{I} = \left\{I_1, I_2, \ldots, I_t, N_{n_1}, N_{n_2}, \ldots, N_{n_s}, M_{n_1}, \ldots, M_{n_s}, M_{n'_{r+1}}, \ldots, M_{n'_q}\right\}.$$

Conversely, any element of $\mathscr{I}$ contains a prime factor of I, hence it contains I. It follows that the radical of I is the intersection of all the elements of $\mathscr{I}$. By Lemma 5.7 this has the desired form. ∎

In the last part of the section we are concerned with the relations between the lattice operations and the unique decomposition of ideals into prime ideal factors. Precisely, having the prime factorizations for two ideals of U (or, equivalently, having the expressions (7) or (8) for two ideals), we want to determine whether or not they are included one in another. We also need to figure out the prime ideal decompositions for the intersection and the sum of two given ideals. The next three results are devoted to these aspects of the problem. But before we state them, it is useful to write down an explicit relation between the expression given in Theorem 3.7 and the prime ideal factorization given in Theorem 5.6.

Remark 5.9. Consider I to be a proper ideal in U and write

$$I = (x^m P(h), P(h)Q(h)) = I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_1^{b_1} N_2^{b_2} \cdots N_n^{b_n} M_1^{c_1} M_2^{c_2} \cdots M_n^{c_n}$$

as in Theorems 3.7 and 5.6. If $I_k = (P_k(h))$ with P_k monic, then

$$P(h) = P_1(h)^{a_1} P_2(h)^{a_2} \cdots P_t(h)^{a_t} \times (h - 1^2 + 1)^{b_1} (h - 2^2 + 1)^{b_2} \cdots (h - n^2 + 1)^{b_n}$$

and

$$Q(h) = (h - 1^2 + 1)^{c_1} (h - 2^2 + 1)^{c_2} \cdots (h - n^2 + 1)^{c_n}.$$

PROPOSITION 5.10. *With the usual notation consider*

$$I = I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_1^{b_1} N_2^{b_2} \cdots N_n^{b_n} M_1^{c_1} M_2^{c_2} \cdots M_n^{c_n}$$

and

$$I' = I_1^{a'_1} I_2^{a'_2} \cdots I_t^{a'_t} N_1^{b'_1} N_2^{b'_2} \cdots N_n^{b'_n} M_1^{c'_1} M_2^{c'_2} \cdots M_n^{c'_n},$$

where $a_i, a'_i, b_j, b'_j, c_j, c'_j \geq 0$, *and* $c_j, c'_j \leq 1$. *Then* $I \subseteq I'$ *if and only if the following conditions are satisfied*:

(i) $a_j \geq a'_j$ *for all* $1 \leq j \leq t$,

(ii) $b_i \geq b'_i$ *for all* $1 \leq i \leq n$, *and*

(iii) $b_i + c_i \geq b'_i + c'_i$ *for all* $1 \leq i \leq n$.

Proof. Write $I = (x^m P(h), P(h)Q(h))$ and $I' = (x^{m'} P'(h), P'(h)Q'(h))$ as in Theorem 3.7 and recall that from Proposition 3.11 we have

$$I \subseteq I' \qquad \text{if and only if} \qquad P' | P \text{ and } P'Q' | PQ.$$

Using Remark 5.9, the latter is equivalent to (i), (ii), and (iii). ∎

PROPOSITION 5.11. *Let I and I' be two proper ideals of U. Write*

$$I = I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_1^{b_1} N_2^{b_2} \cdots N_n^{b_n} M_1^{c_1} M_2^{c_2} \cdots M_n^{c_n}$$

and

$$I' = I_1^{a'_1} I_2^{a'_2} \cdots I_t^{a'_t} N_1^{b'_1} N_2^{b'_2} \cdots N_n^{b'_n} M_1^{c'_1} M_2^{c'_2} \cdots M_n^{c'_n},$$

where the exponents are nonnegative integers and $c_i, c'_i \in \{0, 1\}$. *Then*

$$I \cap I' = I'' = I_1^{a''_1} I_2^{a''_2} \cdots I_t^{a''_t} N_1^{b''_1} N_2^{b''_2} \cdots N_n^{b''_n} M_1^{c''_1} M_2^{c''_2} \cdots M_n^{c''_n},$$

with

$$a_i'' = \max(a_i, a_i') \qquad \text{for all } 1 \le i \le t,$$

$$b_i'' = \max(b_i, b_i') \qquad \text{for all } 1 \le i \le n,$$

$$b_i'' + c_i'' = \max(b_i + c_i, b_i' + c_i') \qquad \text{for all } 1 \le i \le n.$$

Proof. Write $I = (x^m P, PQ)$, $I' = (x^{m'} P', P'Q')$, and $I'' = (x^{m''} P'', P''Q'')$ as in Theorem 3.7. Now Proposition 3.10 yields $P'' = \operatorname{lcm}(P, P')$ and $P''Q'' = \operatorname{lcm}(PQ, P'Q')$. By Remark 5.9, these are equivalent to $a_i'' = \max(a_i, a_i')$, $b_i'' = \max(b_i, b_i')$, and $b_i'' + c_i'' = \max(b_i + c_i, b_i' + c_i')$. ∎

The next theorem gives the *greatest common divisor* expression for the sum of two ideals of U.

PROPOSITION 5.12. *Let I and I' be two proper ideals of U. Write*

$$I = I_1^{a_1} I_2^{a_2} \cdots I_t^{a_t} N_1^{b_1} N_2^{b_2} \cdots N_n^{b_n} M_1^{c_1} M_2^{c_2} \cdots M_n^{c_n}$$

and

$$I' = I_1^{a_1'} I_2^{a_2'} \cdots I_t^{a_t'} N_1^{b_1'} N_2^{b_2'} \cdots N_n^{b_n'} M_1^{c_1'} M_2^{c_2'} \cdots M_n^{c_n'},$$

where the exponents are nonnegative integers and the c's are either 0 *or* 1. *Then*

$$I + I' = I'' = I_1^{a_1''} I_2^{a_2''} \cdots I_t^{a_t''} N_1^{b_1''} N_2^{b_2''} \cdots N_n^{b_n''} M_1^{c_1''} M_2^{c_2''} \cdots M_n^{c_n''},$$

where

$$a_i'' = \min(a_i, a_i') \qquad \text{for all } 1 \le i \le t,$$

$$b_i'' = \min(b_i, b_i') \qquad \text{for all } 1 \le i \le n,$$

$$b_i'' + c_i'' = \min(b_i + c_i, b_i' + c_i') \qquad \text{for all } 1 \le i \le n.$$

Proof. The proof is totally similar to the one for Proposition 5.11. It uses Proposition 3.9 instead of Proposition 3.10. ∎

We close this paper with two results concerning infinite intersections of two-sided ideals in U.

PROPOSITION 5.13. *If $\mathscr{I}$ is an infinite set of distinct two-sided ideals in U, then*

$$\bigcap_{I \in \mathscr{I}} I = 0.$$

Proof. This is equivalent to showing that any nonzero ideal is contained in only finitely many other ideals. For this, let P_1, $P_2, \dots, P_s$, M_1, $M_2, \dots, M_r$ be the finitely many prime ideals containing I, where P_i is of height one and M_j of height two for $1 \le i \le s$ and $1 \le j \le r$. If J contains I, then the prime factors of J contain I so they are among this set. Say $J = \prod P_i^{a_i} \prod M_j^{c_j}$. Of course $c_j = 0$ or 1 and $a_i \le P_i$-exponent in I by Proposition 5.10, so $J = \prod P_i^{a_i} \prod M_j^{c_j}$ has only finitely many possibilities. ∎

COROLLARY 5.14. *If I is an ideal of U and $J = \bigcap_{i=1}^{\infty} I^i$, then either $J = 0$ or $J = I$. Moreover, $J = I$ if and only if the prime ideal factorization for I involves only height two primes in U.*

Proof. If $I = M_{n_1} M_{n_2} \cdots M_{n_r}$ is a product of height two primes of U, then

$$I^2 = M_{n_1}^2 M_{n_2}^2 \cdots M_{n_r}^2 = M_{n_1} M_{n_2} \cdots M_{n_r} = I$$

by Remark 5.5. Consequently, $I^i = I$ for all $i \ge 1$ and clearly $J = I$. If I has a height one prime factor $L = (P(h))$, then $I \subseteq L$ and have

$$J = \bigcap_{i=1}^{\infty} I^i \subseteq \bigcap_{i=1}^{\infty} L^i = 0$$

by Proposition 5.13. Conclude that $J = 0$. ∎

REFERENCES

1. V. V. Bavula, Description of two-sided ideals in a class of noncommutative rings, I, *Ukrainian Math. J.* **45**, No. 2 (1993), 223–234.
2. V. V. Bavula, Description of two-sided ideals in a class of noncommutative rings, II, *Ukrainian Math. J.* **45**, No. 3 (1993), 329–334.
3. S. Catoiu, Ideals of enveloping algebras, Ph.D. thesis, University of Wisconsin-Madison, 1997.
4. S. Catoiu, Ideals in tensor powers of the enveloping algebra $U(sl_2)$, preprint.
5. S. Catoiu, Prime ideals of the enveloping algebra $U(sl_3)$, preprint.
6. J. Dixmier, Idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple complexe, *C. R. Acad. Sci. Paris Sér. A* **271** (1970), 134–136.
7. J. Dixmier, Idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple complexe, *C. R. Acad. Sci. Paris Sér. A* **272** (1971), 1628–1630.
8. J. Dixmier, Quotients simples de l'algèbre enveloppante de sl_2, *J. Algebra* **24** (1973), 551–564.
9. J. Dixmier, "Enveloping Algebras," North-Holland, New York, 1977.
10. J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Graduate Texts in Math., Vol. 9, Springer-Verlag, New York/Berlin, 1972.
11. N. Jacobson, "Lie Algebras," Dover, New York, 1979.
12. J. C. McConnell and J. C. Robson, "Noncommutative Noetherian Rings," Wiley–Interscience, New York, 1987.
13. Y. Nouazé and P. Gabriel, Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente, *J. Algebra* **6** (1967), 77–99.
14. D. S. Passman, "A Course in Ring Theory," Wadsworth, Belmont, CA, 1991.
15. S. P. Smith, The primitive factor rings of the enveloping algebra of $sl(2, C)$, *Proc. London Math. Soc.* **24** (1981), 97–108.